Chapter 6

Analysis and Design of Control Systems in the Time Domain

6.1 Concepts of feedback control

Given a system, we can classify it as an open loop or a closed loop depends on the usage of the feedback.

An open loop system adjust the flow of energy according to a prescribed schedule. We may collect the information of the system and the environment first, then construct the corresponding control. However, once the control is set up and the system is running, we will not change it in-line.

On the other hand, a closed-loop or feedback control system will measure the output, and use the real-time information to adjust the system’s control and action to achieve the intended goal.

The open loop control is relatively easy and inexpensive, but is limited to situations where events are quite predictable. For example, when a plane is landing, the pilot can set up an open loop control program to auto guide the plane to follow a pre-designed trajectory. This may be fine when there is no wind or the wind is constant at the time. However, if there is a gusty wind, the plane may be blown away from the intended trajectory. In this case, we need to apply the closed loop control make sure the plane will follow the desired trajectory. The sensor will detect the error between the real position and the desired one, then adjust the input accordingly.

This can be further illustrated by the following block diagram. Suppose the input of the system is $u(t)$, the output is $x(t)$, and the transfer function is $T(s) = \frac{X(s)}{U(s)}$. We want to obtain the desired output $x_r$, or $X_r(s)$. Using the open-loop control, we calculate the corresponding controller $U(s)$, such that $U(s) \cdot T(s) = X_r(s)$. However, if the transfer function is not perfect or there is any disturbance of the control input, the result will not be accurate. Using the closed-loop control, we detect the output, compare it with the desired value, then apply some control law in the controller to calculate the corresponding control input in real time. Even the system model is not perfect or
there is disturbances, the output can still reach the desired value and maintain it.

### 6.2 Block Diagram

First, let us review the transfer function. Consider the model

\[ \tau \dot{y} + y = b f(t) \]

and assume \( y(0) = 0 \). Then the Laplace transform of the equation is

\[ \tau SY(s) + Y(s) = bF(s) \]

Therefore, the ratio \( Y(s)/F(s) \) is

\[ \frac{Y(s)}{F(s)} = \frac{b}{\tau s + 1} \]

We call the ratio \( Y(s)/F(s) \) the “transfer function” of the model. In general, the transfer function of a linear system is the ratio of the Laplace transform of the output to the transfer function of the input, with the initial condition assumed to be zero, i.e., the ratio of the transforms of the forced response and the input. The transfer function describes how the forced response results from the characteristics of the system itself, instead of the characteristics of the input. E.g., we can obtain the time constant and the natural oscillation frequencies from the roots of the denominator, or the characteristic polynomial.

If we write down the transfer function as \( T(s) \), then we can write the forced response as

\[ Y(s) = T(s) F(s) \]

Since the transfer function can be considered to describe the property of the system, we can put it into a block diagram to demonstrate the relationship between the input and output.

**Example 6.1** Given a first order dynamic system such as a rocket sled, its dynamic equation is:

\[ m \dot{v} + c v = f(t) \]

The transfer function can be written as:

\[ T(s) = \frac{V(s)}{F(S)} = \frac{1}{ms + c} \]

Therefore, we can draw the block diagram as:

![Block Diagram](image)

The three important elements of the block diagram include:
1. The block: the transfer function of the system
2. The arrow in: the transform of the input
3. The arrow out: the transform of the output

The block diagram is a graphical representation of the cause-and-effect relations for one particular system. Besides the general transfer function, there may be some other types of elements in the figure of a block diagram:

- **Multiplier:** \( Y(s) = KX(s) \)

- **Summer/Comparator:** \( Y(s) = X_1(s) \pm X_2(s) \)

- **Takeoff point:** \( Y(s) = X(s) \)

Also, we should note that the block diagram for a given system is not unique. For example, in the previous rocket sled example with zero initial condition, we can rearrange the equation for the dynamic model as:

\[
msV(s) = F(s) - cV(s) \\
V(s) = \frac{1}{ms}(F(s) - cV(s))
\]

which correspond to the diagram
6.2.1 Block diagram reduction

We can use the block diagram reduction to simplify the block diagram or to check whether different forms of the block diagram are equivalent.

Here are some common reduction formulas:

1. Series or cascaded elements

\[
\begin{align*}
Y(s) & \xrightarrow{G_1(s)} W(s) \xrightarrow{G_2(s)} X(s) \\
W(s) &= G_1(s) Y(s) \\
X(s) &= G_2(s) W(s) \\
\Rightarrow X(s) &= G_1(s) G_2(s) Y(s)
\end{align*}
\]

2. Feedback loop:

\[
\begin{align*}
E(s) &= Y(s) + H(s)X(s) \\
X(s) &= G(s)E(s) \\
\Rightarrow Y(s) &= \frac{G(s)}{1 + G(s) H(s)} V(s)
\end{align*}
\]

3. Relocated summer:
6.3. **TWO-POSITION CONTROL**

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\[
E(s) = G_1(s) Y(s) + G_2(s) W(s)
\]

\[
Y(s) = G_3(s) E(s)
\]

\[\Rightarrow\]

\[
Y(s) = G_1(s) G_3(s) Y(s) + G_2(s) G_3(s) W(s)
\]

4. Relocated take-off point:

**Example 6.2** Reduce the following block diagram.

6.3 **Two-Position Control**

The automatic controllers use pre-designed **control law** to act on the error signal to produce the control signal. The control law is also called the **control logic** or the **control algorithm**. Given a control system, the error signal always results from either a change in command or a disturbance. The general function of the controller is to keep the controlled variable near its desired value when these occur.
Two-position controller is one of the most simple and widely used type of controllers. For example, the home thermostats use the \textit{on-off} controller. See the following figure.

![Block diagram of a thermostat system for temperature control.](image)

Fig. 6.1: Block diagram of a thermostat system for temperature control.

The controller output is either on or off. Suppose the current temperature of the room is 60° and we set thermostat with the desired temperature to be 70°. Then the controller turn on, heat the room to 70° and turned off. Because the heat exchange between the room and the outdoor environment, the room temperature will drop to below 70°. This will trigger the controller to turn the heater on again. And this cycle will keep going on and on.

A practical issue is that we don’t want the system to turn on and off too often. Therefore, we set the controlled variable to cycle with an amplitude that depends on the width of the neutral zone or gap. Then, in a real system, the temperature will not be a constant, but oscillate about the desired value. The amplitude of the oscillation depends on the setting of the neutral zone. Since we are not very sensitive to the small change of the temperature, say a couple of degrees, this control is suffice for such system.

![Neutral zone of the on-off controller.](image)

Fig. 6.2: Neutral zone of the on-off controller.

Another type of the two-position control is the \textit{bang-bang} control. Its switching diagram is shown in the figure. An ideal bang-bang controller will have two control values. However, for a real one, it always comes with a deal zone.

![Bang-bang controller.](image)

Fig. 6.3: Bang-bang controller (a) Ideal case (b) With dead zone.

6.4 Transient-Response Specifications

1. Delay time $t_d$: time to reach 50%.

2. Rise time $t_r$: time required to rise from 10% to 90% or according to other specification, from 0% to 100%, or 5% to 95%.
3. Peak time $t_p$: the time the maximum overshoot occurs.

4. Maximum (percentage) overshoot: the maximum deviation of the output $x(t_p)$ above its steady state value $x_{ss}$. The maximum percentage overshoot is $\frac{x(t_p) - x_{ss}}{x_{ss}} \times 100\%$.

5. Settling time $t_s$: the time required for the oscillations to reach and stay within some specified small percentage (e.g. 2%) of the steady state value.

We note that it is often preferable that the transient response be sufficiently fast as well as reasonably damped. For a second order system, an optimal damping ratio $\zeta$ is often chosen between 0.4 and 0.8. Too small $\zeta$ always yields excessive overshoot in the transient response, while too large $\zeta$ will make the response sluggish. Also, we observe that the maximum overshoot and the rise time conflict with each other. In other words, both the maximum overshoot and the rise time cannot be made smaller simultaneously. If one is made smaller, the other necessarily becomes larger.
6.5 PID Control

A widely used controller in industry is the PID controller 6.5. It consists of three parts: Proportional control, Integral control, and Derivative control. A real controller can be either one or a combination of them. However, in practical, the most commonly used sets are P, PI, PD, and PID.

Let the input of the controller be the error between the desired value and the feedback of the system output (denoted by $e(t)$ or $E(s)$ in s-domain), while the output be the system input $u(t)$ or $U(s)$, then the controller transfer function will be $T(s) = \frac{U(s)}{E(s)}$.

- **Proportional control**: $T(s) = K_p$ where $K_p$ is a constant and always being called proportional control gain. The corresponding function in time domain is:
  \[ u(t) = K_p e(t) = K_p (x_r(t) - x(t)) \quad (6.1) \]
  The proportional control means that the control input will decrease when the output is close to the desired value. On the other word, the bigger the error, the greater the control input to pull it to the desired value. In general, higher proportional gain leads to faster system response time.

- **Derivative control**: $T(s) = K_D s$ where $K_D$ is the derivative control gain. Again, in the time domain, the control equation is:
  \[ u(t) = K_D \frac{de(t)}{dt} \quad (6.2) \]
  The derivative control is indeed the proportional control to the first order derivative of the system. It controls the rate the output reaching the desired value. It always used to damp the oscillation of the system.

- **Integral control**: $T(s) = K_I / s$ where $K_I$ is the integral control gain. In time domain, the controller is:
  \[ u(t) = K_I \int_0^t e(t) dt \quad (6.3) \]
  The integral control represents the accumulated result the error. When we just apply the PD control, the steady state of the system may away from the desired value because of the disturbance or the particular PD gain value. Therefore, we can take advantage of the integral control to eliminate the steady state error. However, from the transfer function of the control law, we can see that it will increase the order of the system, and may make an otherwise stable system unstable.
### 6.5. PID CONTROL

#### 6.5.1 Proportional control of a first order system

Before we study the proportional control, let’s look at a simple closed-loop control system. It is a simplified version of the speed control for a DC motor. Suppose the transfer function of the motor is \( \frac{1}{Is+c} \), we want to control the speed of the motor, \( \omega(t) \). The input is the desired speed, or the reference point, \( \omega_r(t) \). We want the output \( \omega(t) \) to follow the change of \( \omega_r(t) \). The block diagram of the system can be shown as following:

Therefore, we can find the equivalent transfer function:

\[
\frac{\Omega(s)}{\Omega_r(s)} = \frac{\frac{1}{I} \frac{1}{s+c}}{1 + \frac{1}{Is+c}} = \frac{1}{Is + c + 1}
\]

Suppose we have a change in the desired value of the output. For simplicity, we simulate it with a unit step input \( 1(t) \) since the system is linear and we can always scale it up or down. Its Laplace transform is \( \Omega_r(s) = \frac{1}{s} \). Therefore, the output becomes

\[
\Omega(s) = \frac{1}{s} \frac{1}{Is + c + 1} = \frac{1}{c + 1} \left( \frac{1}{s} - \frac{1}{s + \frac{c+1}{I}} \right)
\]

Then in the time domain, the speed of the motor becomes

\[
\omega(t) = \frac{1}{c + 1} \left( 1 - e^{-\frac{c+1}{I}t} \right), \quad t \geq 0
\]

With the steady state as

\[
\omega_{ss} = \frac{1}{c + 1}
\]

A more convenient way to find the steady state is

\[
\omega_{ss} = \lim_{s \to 0} s \cdot \frac{1}{Is + c + 1} \frac{1}{s} = \lim_{s \to 0} \frac{1}{Is + c + 1} = \frac{1}{c + 1}
\]

Since \( c \neq 0 \), \( \omega_{ss} = \frac{1}{c+1} \neq 1 \). That is, the output will always be a fraction of the input, and this value is determined by the system itself. In order to control the system to closely follow the input, we add a proportional control to the system. One way to add the gain is after the output, but it needs extra mechanical systems such as a set of gears, which is difficult to implement. Another way is to add the gain to the difference, as shown in the following figure. We add an extra control block as the proportional gain \( K_p \).

Therefore, the transfer function is:

\[
\frac{\Omega(s)}{\Omega_r(s)} = \frac{\frac{K_p}{I} \frac{1}{s+c}}{1 + \frac{K_p}{Is+c}} = \frac{K_p}{Is + c + K_p} = \frac{\frac{K_p}{c+K_p} \frac{1}{s}}{s + \frac{1}{c+K_p}}
\]
Suppose we have a change in the desired value of the output. We simulate it with a unit step input $1(t)$ since the system is linear and we can always scale it up or down. Its Laplace transform is $\Omega_r(s) = 1/s$. Therefore, the output becomes

$$\Omega(s) = \frac{K_p}{Is + c + K_p} \frac{1}{s}$$

(6.5)

The steady state value will be

$$\omega_{ss} = \lim_{s \to 0} s \cdot \frac{K_p}{Is + c + K_p} \frac{1}{s} = \frac{K_p}{c + K_p} < 1$$

(6.6)

We can see that the final steady state value will always less than the desired one. However, if we set $K_p \gg c$, $\omega_{ss}$ will very close to the 1. The time constant of the transfer function is $\tau = \frac{1}{c+K_p}$, and the response can be obtained through partial fraction expansion and inverse Laplace transform:

$$\omega(t) = \frac{K_p}{c + K_p} \left[ 1 - e^{-\frac{c+K_p}{T}t} \right]$$

(6.7)

Also, we assume there is a disturbance $D(s)$. Then the transfer function becomes

$$\frac{\Omega(s)}{D(s)} = \frac{-\frac{1}{Is + c}}{1 - \frac{K_p}{Is + c}} = \frac{-1}{Is + c + K_p} = \frac{-\frac{1}{c+K_p}}{\frac{1}{c+K_p}s + 1}$$

(6.8)
Likewise, if we model the disturbance as a unit step function too, i.e., \( D(s) = 1/s \). The corresponding response is

\[
\Omega(s) = \frac{-1}{Is + c + K_p s} \frac{1}{s} \quad (6.9)
\]

The steady state value is

\[
\omega_{ss} = \lim_{s \to 0} s \cdot \frac{-1}{Is + c + K_p s} = \frac{-1}{c + K_p} \quad (6.10)
\]

while the system response is

\[
\omega(t) = -\frac{1}{c + K_p} \left[ 1 - e^{-\frac{c+K_p}{I}t} \right] \quad (6.11)
\]

We note that if \( c + K_p \gg 1 \), the effect of the disturbance on the output will be small. Further, the response for the disturbance can not stand alone. It should be always added to Eq. 6.7 for actual result.

The purpose of the control system is that we want the output value to follow the required one, or so called the command input. Meanwhile, the disturbance should be rejected, that is, has zero response. However, it is impossible to obtain such an ideal system. In many applications, we just require that the steady state is what we desired. In many other cases, we will add more requirements, such as the response time, the overshoot percentage, etc. Different techniques had developed for different extra requirements, and we should pick the best one for one particular application.

Now, we suppose the parameter values are \( I = 2 \) and \( c = 3 \). Again, both reference input and disturbance are unit steps. We want to find out the smallest value of the gain \( K_p \) such that the steady state offset error will be no greater than 0.2 for the command input.

The steady state offset error is

\[
\omega_r - \omega_{ss} = 1 - \frac{K_p}{3 + K_p} \leq 0.2 \quad (6.12)
\]

Therefore, we have

\[
0.8 \leq \frac{K_p}{3 + K_p} \Rightarrow 2.4 + 0.8K_p \leq K_p \Rightarrow K_p \geq 12 \quad (6.13)
\]

With such control gain, the steady state disturbance response will be

\[
\frac{-1}{3 + K_p} = \frac{-1}{3 + 12} = \frac{-1}{15} = -0.067 \quad (6.14)
\]

Then the total steady state response will be

\[
(1 - 0.2) + (-0.067) = 0.733 \quad (6.15)
\]

### 6.5.2 PI Control of a First order plant

The offset error that occurs with proportional control is a result of the system reaching an equilibrium in which the control signal no longer changes. This allows a constant error to exist. In order to eliminate this steady state error, we add the integral control to the controller. That is, we let the steady state error to accumulate a control signal, which can be used to determine
the existence and level of the offset. Therefore, we develop a new control law called the integral control:

\[
\frac{U(s)}{E(s)} = \frac{K_I}{s}
\]  

(6.16)

where \(U(s)\) is the control signal and \(K_I\) is called integral gain.

In general, since the integral control will increase one degree of the system and may bring unwanted transient behavior to the system, we don’t use it by itself. Instead, we often use proportional-integral (PI) control:

\[
\frac{U(s)}{E(s)} = K_p + \frac{K_I}{s} = K_p(1 + \frac{1}{T_1 s})
\]  

(6.17)

where \(T_1\) is the reset time. The reset time is the time required for the integral action signal to equal that of the proportional term if a constant error exists. The integral term does not react the error instantaneously, but continues to correct. If the system is not designed well, it may lead to the oscillation of the system.

Let’s look at the same system with PI controller.

The transfer function becomes

\[
\frac{\Omega(s)}{\Omega_r(s)} = \frac{(K_p + K_I)}{Is + c} = \frac{K_p s + K_I}{Is^2 + (c + K_p)s + K_I}
\]  

(6.18)

\[
\frac{\Omega(s)}{D(s)} = \frac{-\frac{1}{Is + c}}{1 - \frac{(K_p + K_I)}{Is + c}} = -\frac{s}{Is^2 + (c + K_p)s + K_I}
\]  

(6.19)

Note that by introducing an integral controller, we changed the first order system to a second order system.

For a unit step command input, the response is

\[
\omega_{ss} = \lim_{s \to 0} s \cdot \frac{K_p s + K_I}{Is^2 + (c + K_p)s + K_I} \cdot \frac{1}{s} = \frac{K_I}{K_I} = 1
\]  

(6.20)

while the response for the unit step disturbance becomes

\[
\omega_{ss} = \lim_{s \to 0} s \cdot \frac{-s}{Is^2 + (c + K_p)s + K_I} \cdot \frac{1}{s} = \frac{0}{K_I} = 0
\]  

(6.21)

However, by applying the integral control, the order of the system increases one. The first order system becomes the second order one. The characteristic equation becomes

\[
Is^2 + (c + K_p)s + K_I = 0
\]  

(6.22)
It is possible that the system will oscillate. So, we need to choose the gains carefully. The damping ratio is

$$\zeta = \frac{c + K_p}{2\sqrt{IK_I}}$$  \hspace{1cm} (6.23)

If we choose it to be underdamped or critically damped ($\zeta \leq 1$), then we can define the time constant as:

$$\tau = \frac{2I}{c + K_p}$$  \hspace{1cm} (6.24)

From different requirements of $\zeta$ and $\tau$, we can pick up the best combination of $K_p$ and $K_I$.

**Example 6.3 Computing PI control gains**

Suppose $I = 4$, $c = 4$, and we require that the time constant $\tau = 0.2$. We want to choose the gains for $\zeta = 0.707$ and $\zeta = 1$.

Substitute the values of $I$ and $c$ to the command transfer function (Eq. 6.20), then we have

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K_p s + K_I}{4s^2 + (4 + K_p)s + K_I}$$ \hspace{1cm} (6.25)

Since $\zeta \leq 1$: The time constant is

$$\tau = \frac{2 \cdot 4}{4 + K_p} = \frac{8}{4 + K_p} = 0.2$$ \hspace{1cm} (6.26)

Therefore, we can solve for the proportional control gain

$$K_p = 36$$ \hspace{1cm} (6.27)

From the definition of damping ratio, we can get

$$\zeta = \frac{c + K_p}{2\sqrt{IK_I}} = \frac{40}{2\sqrt{4K_I}} = \frac{10}{\sqrt{K_I}}$$ \hspace{1cm} (6.28)

Then

$$\zeta = 0.707 \Rightarrow K_I = 200$$ \hspace{1cm} (6.29)

$$\zeta = 1 \Rightarrow K_I = 100$$ \hspace{1cm} (6.30)
6.5.3 Proportional Control of a Second-Order Plant

We first give a definition of stability: A constant-coefficient linear model is

- **stable** if, and only if, all of its characteristic roots have negative real parts. For example, \( \dot{x} + x = 0 \) has the root \( s = -1 \). If the initial condition is \( x(0) = 1 \), then its free response is \( x(t) = e^{-t} \), which is stable;

- **neutrally stable** if one or more roots have a zero real part, and the remaining roots have negative real parts. E.g., \( \ddot{x} + 1000x = 0 \) has the roots \( s = j \pm 31.62 \). If the initial condition is \( x(0) = 1, \dot{x} = 0 \), then its free response is \( x(t) = \sin(31.62t + \frac{\pi}{2}) \), which is neutrally stable;

- **unstable** if any root has a positive real part. E.g., \( \ddot{x} - x = 0 \) has the roots \( s = \pm 1 \). If the initial condition is \( x(0) = 1, \dot{x} = 0 \), then its free response is \( x(t) = \frac{1}{2}(e^{t} + e^{-t}) \), which is unstable.

Clearly, we generally want the system to be stable. For the neutrally stable or unstable system, we can use the control algorithm to stabilize it. Suppose we want to control the angle instead of the speed of the motor, then the feedback controlled system is as following one:

The transfer functions are:

\[
\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_p}{I s^2 + cs + K_p} \quad (6.31)
\]

\[
\frac{\Theta(s)}{D(s)} = \frac{-1}{I s^2 + cs + K_p} \quad (6.32)
\]

while the characteristic equation is

\[ I s^2 + cs + K_p = 0 \quad (6.33) \]

One thing we should note is that if suppose \( I, c > 0 \), the original plant is neutrally stable from its characteristic equation \( s(I s + c) = 0 \). However, if we let \( K_p > 0 \), the system becomes stable. That is, by introducing the controller, the stability of a system can be changed.

For steady state response, if the command input is a unit step \( \Theta_r(s) = \frac{1}{s} \), then the output is

\[
\Theta(s) = \lim_{s \to 0} s \cdot \frac{K_p}{I s^2 + cs + K_p} \cdot \frac{1}{s} = 1 \quad (6.34)
\]

That is, for this particular second order system, the steady state error will be 0. However, we note that it is not hold for a general system such as \( \frac{1}{I s^2 + cs + k} \). The corresponding steady state error is \( \frac{k}{k+K_p} \).
6.5. PID CONTROL

If the disturbance is also a unit step \( D(s) = \frac{1}{s} \). Its corresponding response represented in output is

\[
\Theta(s) = \lim_{s \to 0} s \cdot \frac{-1}{Is^2 + cs + K_p s} \cdot \frac{1}{K_p} = -\frac{1}{K_p}
\]

(6.35)

The transient behavior is indicated by the damping ratio:

\[
\zeta = \frac{c}{2\sqrt{IK_p}}
\]

(6.36)

From the steady state response to the disturbance, we can see that the larger \( K_p \), the less the error. However, if \( K_p \) is too large, the damping ratio \( \zeta \) will be very small. A small \( \zeta \) means the transient behavior will be very oscillatory and the overshoot will be large. Therefore, the final choice of \( K_p \) will be a compromise between the requirement of steady state error and the oscillation. Another choice will be adding a derivative controller.

6.5.4 PD Control of a Second-order Plant

Following is the same system with a PD controller.

The transfer functions are

\[
\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_p + K_ds}{Is^2 + (c + K_d)s + K_p}
\]

\[
\frac{\Theta(s)}{D(s)} = \frac{-1}{Is^2 + (c + K_D)s + K_p}
\]

(6.37)  (6.38)

The steady state response correspond to the unit step command input and the unit step disturbance are still the same

\[
\Theta(s) = \lim_{s \to 0} s \cdot \frac{K_p + K_ds}{Is^2 + (c + K_D)s + K_p} \cdot \frac{1}{s} = 1
\]

(6.39)

\[
\Theta(s) = \lim_{s \to 0} s \cdot \frac{-1}{Is^2 + (c + K_D)s + K_p} \cdot \frac{1}{s} = -\frac{1}{K_p}
\]

(6.40)

However, the transient response will be modified by the proportional gain \( K_D \)

\[
\zeta = \frac{c + K_D}{2\sqrt{IK_p}}
\]

(6.41)

If we choose \( \zeta < 1 \), the time constant is

\[
\tau = \frac{2I}{c + K_D}
\]

(6.42)
Therefore, we can choose a large proportional control gain $K_p$ to reduce the steady state error, while use a large derivative control gain $K_D$ to attenuate the oscillation and overshooting in transient response.

**Example 6.4 Computing PD control gains.**

Suppose $I = 10$ and $c = 3$. The performance specifications require that $\tau = 1$ and $\zeta = 0.707$.

Since

$$\tau = \frac{2I}{c + K_D} = \frac{20}{3 + K_D} = 1$$

we have $K_D = 17$. Then from

$$\zeta = \frac{c + K_D}{2\sqrt{TK_p}} = \frac{20}{2\sqrt{10K_p}} = \sqrt{\frac{10}{K_p}} = 0.707$$

we can get $K_p = 20$.

**Method 2:** Another method is to determine the roots first. Since $\tau = \frac{1}{\zeta\omega_n}$, we have

$$\omega_n = \frac{1}{\tau\zeta} = \sqrt{2}$$

Then

$$s = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} = -1 \pm i\sqrt{2}\sqrt{1 - \frac{1}{2}} = -1 \pm i$$

Then the characteristic equation is

$$(s + 1 - i)(s + 1 + i) = (s + 1)^2 + 1 = s^2 + 2s + 2 = 0$$

or

$$10s^2 + 20s + 20 = 0$$

Compare with the characteristic equation of the system

$$Is^2 + (c + K_D)s + K_p = 10s^2 + (3 + K_D)s + K_p = 0$$

Then we can obtain $K_D = 17$ and $K_p = 20$. 