Chapter 3

Modeling of a Mechanical System

3.1 Units

Currently, there are two systems of units: One is the international system (SI) or metric system, the other is the British engineering system (BES) or the English system. We prefer to use the SI system in this course, buy may use either of them whenever necessary. Also, within each system, we define certain base units. In SI system, they are meter, kilogram, and second. In BES, they are foot, pound, and second. Almost all the other units used in mechanical system can be derived from these base units, and are categorized as derived units. In this course, the following basic and derived units are used more often.

Quantity	Length	Mass	Time	Force	Energy	Power
SI	m	kg	s	N=kg-m/s^2	J=N-m	W=N-m/s
BES	ft	slug	s	lb	ft-lb	hp

3.2 Mechanical Elements

3.2.1 Inertia Elements

The inertia elements include masses for translation and moments of inertia for rotation. The mass is usually denoted by m with unit as kg or $slug$. The moment of inertia often represented as J with unit as $kg - m^2$.

3.2.2 Spring Elements

A linear spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force or torque applied to it.

For a translational spring, the relation between the acting force F and the net displacement x is

$$F = kx = k(x_1 - x_2) \tag{3.2.1}$$

where k is a proportionality constant called a *spring constant,* the unit of k is N/m or lb/in.

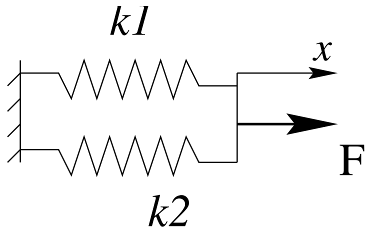
For a rotational motion with a torsional spring, the relation between the acting torque T and the net angular displacement θ is

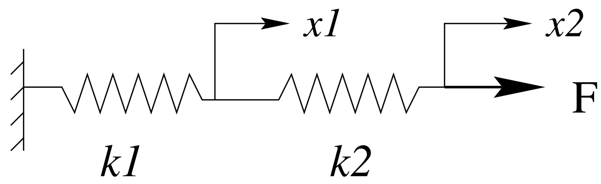
$$T = k\theta = k(\theta_1 - \theta_2) \tag{3.2.2}$$

where k is also a proportionality constant called a *torsional spring constant.*

Note: When a linear spring is overstretched over certain point, it will become nonlinear. In this course, we will assume the springs are always working within its linear limit. Further, although all practical springs have inertia and damping, we assume that the effect of them are negligibly small. Therefore, all the springs in this course will be ideal springs with neither mass nor damping and will obey the linear force-displacement law or linear torque-angular displacement law.

Exercise Given two springs with spring constant k_1 and k_2, obtain the equivalent spring constant k_{eq} for the two springs connected in (1) parallel (2) serial.





Solution: (1) The two springs have same displacement, therefore,

$$k_1 x + k_2 x = F = k_{eq} x,$$

or

$$k_{eq} = k_1 + k_2.$$

(2) The forces on each spring are same, F. However, their displacements are different. Let them be x_1 and x_2. Then,

$$k_1 x_1 = k_2 x_2 = F$$

or

$$x_1 = \frac{F}{k_1} \qquad x_2 = \frac{F}{k_2}$$

Since the total movement is $x = x_1 + x_2$, and we have $F = k_{eq} x$, then we can obtain

$$\frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2}$$

or

$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}.$$

3.2.3 Damping Elements

When the viscosity or drag is not negligible in a system, we often model them with the damping force. The damping force always depends on the relative velocity between the fluid and the surface. In this class, we will only deal with the linear damping force, which is a linear function of the relative velocity.

In engineering, we often encounter three kinds of damping elements: a dashpot, a sliding viscous friction or a journal bearing. The former two can be classified as translational damper. Given the relative velocity, the drag force generated is

$$f = c\dot{x} = c(\dot{x}_1 - \dot{x}_2)$$

where c is a proportionality constant called the **damping coefficient**. On the other hand, the journal bearing is a special kind of rotational damper, with the drag torque:

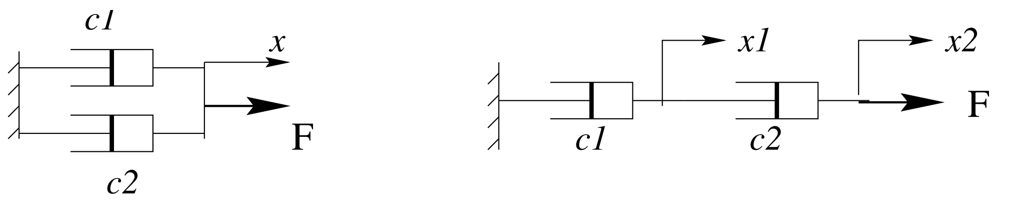
$$T = c \ \ \omega = c(\omega_1 - \omega_2)$$

where $\omega = \dot{\theta}$ is the relative angular velocity.

Like spring elements, we assume the damping elements are only working in their linear range in this course. Also, we consider all the damping elements are idea, that is, with no inertia and spring effects.

When analyze a system, we often use these symbols to denote different damping elements. Note that the direction of the force is always opposite to the direction of the relative motion.

Exercise Given two dampers with damping constant c_1 and c_2, obtain the equivalent damping constant c_{eq} for the two dampers connected in (1) parallel (2) serial.:



Solution: Using the analysis analogous to spring, except that with c instead of k, $\dot{x}$ instead of x, we have (1) in parallel

$$c_{eq} = c_1 + c_2$$

(2) in serial

$$c_{eq} = \frac{c_1 c_2}{c_1 + c_2}$$

Exercise How about putting two masses in parallel and in series?

3.3 Modeling the Simple Translational System

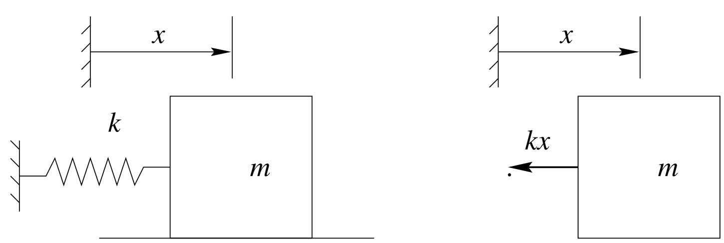
Newton's laws:

1. A particle originally at rest, or moving in a straight line with a constant velocity, will remain that way as long as it is not acted upon by an external force.
2. The time rate of change of momentum equals the external force.

$$\mathbf{f} = \frac{d\,(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{3.3.1}$$

3. Every action is opposed by an equal reaction.

Example 1. A simple horizontal spring-mass system on a frictionless surface.



From the free body diagram (F.B.D.) of the mass, we can see that the only force acting on the mass is the spring force provided that there is no external force. Using Newton's laws, we can easily obtain the dynamic equation:

$$m\ddot{x} = -kx \qquad or \qquad m\ddot{x} + kx = 0$$

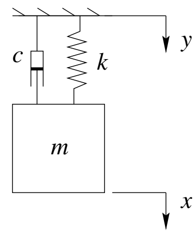
Example 2. A simple vertical spring-mass system.

Now, there is one more external force acting on the mass: the gravity. Therefore, the equation becomes:

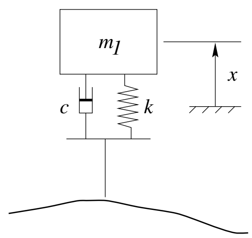
$$m\ddot{y} = -ky + mg$$

However, if we inspect the system more closely, we can see that the gravitational force is always being opposed statically by the equilibrium spring deflection δ, or $mg = k\delta$. If we measure the displacement from the equilibrium position, that is, $x = y - \delta$, or $y = x + \delta$, then the dynamic equation can be simplified to

$$m\ddot{x} = -k(x + \delta) + mg, \qquad or \qquad m\ddot{x} + kx = 0.$$



Example 3. A single-mass model of a car's suspension. When we want to make a preliminary examination of a car's suspension, we always model it as a mass-spring-damper system. Here, we model it with a single mass. The elasticity of both the tire and the suspension spring can be modeled with a spring with spring constant k. The shock absorber is modeled as a translational damper with constant c. If we assume the weight of the car is evenly distributed to the four wheels, and the mass of the wheel, tire and axle are negligible, the mass m of the model is one quarter of the total mass except the moving part. We want to find out the motion of the system.



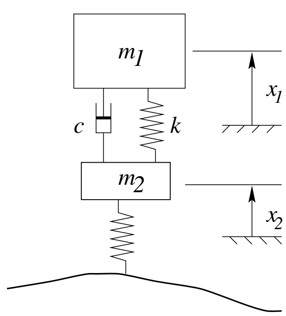
From the free body diagram, we have

$$m\ddot{x} = c(\dot{y} - \dot{x}) - k(y - x)$$

or

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

Example 4. A two-mass model of a suspension. Here we model the system with more complexity. We consider the suspension model with the wheel-tire-axle assembly. As the previous one, the mass m_1 is one-fourth mass of the car body, and m_2 is the mass of the wheel-tire-axle assembly.



From the free body diagram, the equation for m_1 is

$$m_1\ddot{x}_1 = c_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1)$$

or

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_1\dot{x}_2 - k_1x_2 = 0$$

while the equation for m_2 is

$$m_2\ddot{x}_2 = -c_1(\dot{x}_2 - \dot{x}_1) - k_1(x_2 - x_1) + k_2(y - x_2)$$

or

$$m_2\ddot{x}_2 + c_1\dot{x}_2 + (k_1 + k_2)x_2 - c_1\dot{x}_1 - k_1x_1 = k_2y$$

3.4 Modeling the simple rotational system

Rotation about a fixed axis:

$$\mathbf{M} = \frac{d\mathbf{H}}{dt} = I\frac{d\omega}{dt} \tag{3.4.1}$$

where $\mathbf{M}$ is the external applied moments or torque, $\mathbf{H}$ is the angular momentum, I is the body's moment of inertia about the axis, and ω is the vector angular velocity.

Example 5. A model with angular displacement input. Given a shaft supported by a bearing with damping, we want to find out the dynamic equation of the system. The input is the rotation θ_i. Because of the elasticity of the shaft, the angular displacement of the shaft is different, and denoted by θ. The elasticity of the shaft is modeled by a rotational spring k, while the damping constant is c.

From the free body diagram, we have

$$I\ddot{\theta} = k(\theta_i - \theta) - c\dot{\theta}$$

or

$$I\ddot{\theta} + c\dot{\theta} + k\theta = k\theta_1$$

3.5 Work, Energy, and Power

The **work** done in a mechanical system is the dot product of force and distance (or torque and angular displacement), when they are expressed in vector form. Let force be $\vec{F}$ and distance $\vec{x}$, then the work W done by the force along the displacement is

$$W = \vec{F} \cdot \vec{x} \tag{3.5.1}$$

Note that the work is a scalar. It can be positive or negative, but comes with no direction.

The unit of work is joule (J) in SI system,

$$1J = 1N \cdot m$$

Energy is the ability or capacity to do work. It can be as many forms, such as chemical energy, electrical energy, or mechanical energy. In mechanical system, the energy always exists as the **potential energy** or **kinetic energy**, or both.

The potential energy of a mass (m) in mechanical system always dues to the gravitational field. It is defined as

$$U = \int_0^h mgdx = mgh$$

where h is the altitude and g is the local gravity acceleration.

The potential energy of a spring is defined as

$$\begin{aligned} U &= \int_0^x kx dx = \frac{1}{2}kx^2 \qquad \text{(Translational)} \\ U &= \int_0^\theta k\theta d\theta = \frac{1}{2}k\theta^2 \qquad \text{(Rotational)} \end{aligned}$$

Exercise: What are the changes of the potential energy for a mass at different positions, and a spring with different stretch or compression?

Only inertia elements in mechanical systems can store kinetic energy. For a mass m in pure translation, it is

$$T = \frac{1}{2}m\dot{x}^2$$

where $\dot{x}$ is the velocity of the mass. For a moment of inertia J in pure rotation, it is

$$T = \frac{1}{2}J\dot{\theta}^2$$

where $\dot{\theta}$ is the angular velocity of the moment of inertia.

Exercise: How to calculate the change of kinetic energy with different velocity or angular velocity?

Power is the time rate of doing work. Note that there is no time issue when we consider work and energy, while power is the work per unit time.

Conservation of energy: When there is no friction element (such as a damper) in a mechanical system, the total energy will be constant, that is

$$T_1 + U_1 = T_2 + U_2$$

where the subscript $1, 2$ denote two distinct instances.

Exercise: Given the spring-mass system shown by example 1, the motion of the mass can be solved as

$$x(t) = x_0 \cos(\sqrt{\frac{k}{m}}\ t)$$

provided that the initial condition is $x(0) = x_0$ and $\dot{x}_0 = 0$. Verify that the total energy is constant.

Advanced Topic: When there is no energy enter or leave the mechanical system, we often use the Euler-Lagrange Equation to find out the dynamic equation. That is, define the Lagrangian $L = T - U$, the generalized variables q, the motion of the system can be described by

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0 \qquad (3.5.2)$$

To probe further, you are encouraged to read related books or take Advanced Dynamics and other similar courses.